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Abstract: We derive in this paper the transient queue length and the busy period distributions in the single server queue with phase-type (PH) services times and a batch Markovian arrival process (BMAP). These distributions are obtained by the uniformization technique which gives simple recurrence relations. The main advantage of this technique is that it leads to stable algorithms for which the precision of the result can be given in advance.

Key-words: BMAP, busy period, transient queue length, uniformization.

(Résumé : tsvp)

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Longueur transitoire et période d'activité de la file BMAP/PH/1

Résumé : On calcule dans ce rapport les distributions de la longueur transitoire et de la période d'activité d'une file à serveur unique avec des durées de service de type phase (PH) et un processus d'arrivées markovien par groupes (BMAP). Ces distributions sont obtenues par la technique d'uniformisation qui donne des relations de récurrence simples. L'avantage principal de cette technique est qu'elle conduit à des algorithmes stables pour lesquels la précision du résultat peut être fournie à l'avance.

Mots-clé : BMAP, période d'activité, longueur transitoire, uniformisation.

1 Introduction

We consider a single server queue with infinite waiting room for which the arrival process is a Batch Markovian Arrival Process (BMAP) and the service times are given by a Phase-type (PH) distribution. The BMAP has received considerable interest during the last few years. It was first introduced by Neuts [1] as the versatile Markovian point process. It generalizes the Markovian Arrival Process (MAP) introduced by Lucantoni et al. [2]. A tutorial on the BMAP/G/1 queue is presented in [3] and the transient behavior of this queue is analyzed in [4] using numerical inversions of Laplace transforms and generating functions.

We consider here the BMAP/PH/1 queue and we focus on the transient queue length and busy period distributions. Its main advantage with respect to the BMAP/G/1 queue is that the behavior of the BMAP/PH/1 queue is governed by a homogeneous Markov process and so it can be analyzed avoiding the use of Laplace transforms and generating functions inversion algorithms which can lead to severe numerical errors and overflow problems. Moreover the set of PH probability distributions is dense in the set of probability distributions on $(0, \infty)$ [5]. For the analysis of the BMAP/PH/1 queue, we consider the uniformization technique [6]. Its main advantage is that it leads to the analysis of a discrete time Markov chains for which all the required quantities to evaluate are given by recurrence relations involving only additions and multiplications of non negative numbers bounded by one and thus leading to stable algorithms. Moreover, the precision of the results can be given in advance.

The remainder of the paper is organized as follows. In the next section we review some definitions and properties of the BMAP that can also be found with more detail in [3] and we describe the Markov process governing the behavior of the BMAP/PH/1 queue. In section 3 and 4 we derive the recurrence relations used for the computation of the transient queue length distribution and the busy period distribution respectively. We illustrate our method through numerical examples in section 5.

2 The BMAP/PH/1 Queue

2.1 The Batch Markovian Arrival Process

The BMAP is a two dimensional Markov process $\{A(t), J(t)\}$ on the state space $\{(i,j)|i \geq 0,\ 1 \leq j \leq m\}$ with infinitesimal generator given by

$$\begin{bmatrix} D_0 & D_1 & D_2 & D_3 & . & . & . \\ & D_0 & D_1 & D_2 & . & . & . \\ & & D_0 & D_1 & . & . & . \\ & & & D_0 & . & . & . \\ & & & & . & . & . \end{bmatrix}$$

where D_k, $k \geq 0$, are $m \times m$ matrices such that matrix D_0 has negative diagonal elements and non negative off diagonal elements; matrices D_k, $k \geq 1$, have non negative elements and matrix D, defined by

$$D = \sum_{k=0}^{\infty} D_k,$$

is an irreducible infinitesimal generator. We also assume that $D \neq D_0$, which ensures that arrivals will occur.

The variable $A(t)$ counts the number of arrivals during $[0, t[$ and the variable $J(t)$ represents the phase of the arrival process. More detail and results concerning this process can be found in [3] for instance.

2.2 The Queueing Model

We consider a single server queue with a BMAP specified by the sequence $\{D_k,\ k \geq 0\}$. Let the service times be i.i.d. and independent of the arrival process. The service times have a phase-type (PH) distribution [7] with $l+1$ states given by its initial probability distribution (β, β_{l+1}) and infinitesimal generator

$$\begin{bmatrix} T & T_0 \\ 0 & 0 \end{bmatrix}.$$

For the sake of simplicity, we assume that $\beta_{l+1} = 0$. Note that β is a non negative l dimensional row vector satisfying $\beta \mathbf{1} = 1$ where vector $\mathbf{1}$ is a column vector of 1's; its dimension being specified by the context. Matrix T is a $l \times l$ matrix and T_0 is a l dimensional column vector satisfying $T_0 = -T\mathbf{1}$.

The process describing the behaviour of the BMAP/PH/1 queue is then a Markov process $X(t)$, over the state space

$$S = \{(0,j) \mid 1 \leq j \leq m\} \bigcup \{(n,j,k) \mid n \geq 1,\ 1 \leq j \leq m,\ 1 \leq k \leq l\}.$$

The couple $(0,j)$ represents the state for which there are no customer in the queue and the arrival process is in phase j. The triple (n,j,k) represents the state for which there are n customers in the queue, and the arrival process is in phase j and the customer being served is in phase k. With this description of the state space, the infinitesimal generator of the process is given by

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & \cdots \\ A_{1,0} & A_1 & A_2 & A_3 & \cdots \\ & A_0 & A_1 & A_2 & \cdots \\ & & A_0 & A_1 & \cdots \\ & & & A_0 & \cdots \\ & & & & \cdots \end{bmatrix}$$

where

$$A_{0,0} = D_0 \text{ and } A_{0,j} = D_j \otimes \beta \text{ for } j \geq 1,$$

$$A_{1,0} = I_m \otimes T_0 \text{ and } A_1 = (D_0 \otimes I_l) + (I_m \otimes T) \text{ and } A_j = D_j \otimes I_l \text{ for } j \geq 2,$$

$$A_0 = I_m \otimes (T_0 \beta),$$

where the matrix I_r denotes the $r \times r$ identity matrix and $\otimes$ denotes the classical Kronecker product.

Let us define $\nu = \sup\{-A(u,u),\ u \in S\}$. It is easy to verify that

$$\nu = \max\{-D_0(j,j),\ j = 1, \ldots, m\} + \max\{-T(k,k),\ k = 1, \ldots, l\}.$$

Using now the uniformization technique [6], we denote by $Z(n)$ the uniformized Markov chain associated to $X(t)$, with $Z(0) = X(0)$. Its transition probability matrix P is given by $P = I + A/\nu$, where I denotes the infinite identity matrix. Matrix P is a stochastic matrix and has the same structure as matrix A with

$$P_{0,0} = I_m + A_{0,0}/\nu \text{ and } P_{0,j} = A_{0,j}/\nu \text{ for } j \geq 1,$$

$$P_{1,0} = A_{1,0}/\nu \text{ and } P_1 = I_{ml} + A_1/\nu \text{ and } P_j = A_j/\nu \text{ for } j \geq 2,$$

$$P_0 = A_0/\nu.$$

Let B_0 be the subset of states defined by $B_0 = \{(0,j) \mid 1 \le j \le m\}$. For $i \ge 1$, we define the subsets

$$B_i = B_0 \bigcup \{(n,j,k) \mid 1 \le n \le i,\ 1 \le j \le m,\ 1 \le k \le l\}.$$

For every $i \ge 0$, B_i represents the states of S corresponding to at most i customers in the queue.

3 The Transient Queue Length

We denote by α the row vector containing the initial probability distribution of the Markov process $X(t)$ and by $\mathbf{1}_{B_i}$, $i \ge 0$, the infinite column vector with the first $|B_i|$ entries equal to 1 and the other equal to 0.

We define $N(t)$ as the number of customers in the queue at time t. The distribution of $N(t)$ is then given by

$$\Pr(N(t) \le i) = \Pr(X(t) \in B_i) = \alpha e^{At}\mathbf{1}_{B_i} = \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}\Pr(Z(n) \in B_i),$$

where $\Pr(Z(n) \in B_i)$ is given by

$$\Pr(Z(n) \in B_i) = \alpha P^n \mathbf{1}_{B_i}.$$

We now derive recurrence relations for the computation of $\Pr(Z(n) \in B_i)$.

For every $n, i \ge 0$ we define the infinite column vector $U_{B_i}^{(n)}$ as $U_{B_i}^{(n)} = P^n \mathbf{1}_{B_i}$. The infinite column vector $U_{B_i}^{(n)}$ can be decomposed into subvectors $U_{h,i}^{(n)}$, $h \ge 0$. The vector $U_{0,i}^{(n)}$ is of dimension m and for $h \ge 1$, $U_{h,i}^{(n)}$ is of dimension ml. With this notation, we have

$$\begin{aligned}
U_{0,i}^{(n)} &= [\Pr(Z(n) \in B_i \mid Z(0) = (0,j))]_{1\le j\le m} \\
U_{h,i}^{(n)} &= [\Pr(Z(n) \in B_i \mid Z(0) = (h,j,k))]_{1\le j\le m, 1\le k\le l} \text{ for } h \ge 1
\end{aligned}$$

It follows that for $h \ge i+n+1$, we have $U_{h,i}^{(n)} = 0$.

For $n = 0$, we have $U_{B_i}^{(0)} = \mathbf{1}_{B_i}$ and for $n \ge 1$, writing $U_{B_i}^{(n)} = PU_{B_i}^{(n-1)}$, we get the following recurrence relations

$$U_{0,i}^{(n)} = \sum_{h=0}^{i+n-1} P_{0,h} U_{h,i}^{(n-1)}$$

$$
\begin{aligned}
U_{1,i}^{(n)} &= P_{1,0} U_{0,i}^{(n-1)} + \sum_{h=1}^{i+n-1} P_{0,h} U_{h,i}^{(n-1)} \qquad (1)\\
U_{r,i}^{(n)} &= \sum_{h=r-1}^{i+n-1} P_{h-r+1} U_{h,i}^{(n-1)}, \qquad \text{for } 2 \le r \le i+n+1.
\end{aligned}
$$

We now decompose the initial probability distribution α as $\alpha = (\alpha_h)_{h\ge 0}$ where the subvector α_h corresponds to the initial probability distribution when there are exactly h customers in queue, that is

$$
\begin{aligned}
\alpha_0 &= [\Pr(Z(0) = (0,j))]_{1\le j\le m}\\
\alpha_h &= [\Pr(Z(0) = (h,j,k))]_{1\le j\le m, 1\le k\le l} \text{ for } h \ge 1
\end{aligned}
$$

Using this notation we get

$$
\Pr(Z(n) \in B_i) = \sum_{h=0}^{n+i} \alpha_h U_{h,i}^{(n)}.
$$

Let ε the desired error tolerance for the computation of $\Pr(N(t) \le i)$, we define the truncation step N as

$$
N = \min \left\{ n \in \mathbb{N} \;\middle|\; \sum_{j=0}^{n} e^{-\nu t} \frac{(\nu t)^j}{j!} \ge 1 - \varepsilon \right\}. \qquad (2)
$$

The distribution of $N(t)$ can then be written as

$$
\Pr(N(t) \le i) = \sum_{n=0}^{N} e^{-\nu t} \frac{(\nu t)^n}{n!} \Pr(Z(n) \in B_i) + e(N),
$$

where $e(N)$ verifies

$$
\begin{aligned}
e(N) &= \sum_{n=N+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \Pr(Z(n) \in B_i)\\
&\le \sum_{n=N+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!}\\
&= 1 - \sum_{n=0}^{N} e^{-\nu t} \frac{(\nu t)^n}{n!}\\
&\le \varepsilon
\end{aligned}
$$

Remark 1: The computation of integer N can be made without any numerical problems even for large values of νt by using the method described in [8].

Remark 2: The truncation level N is in fact a function of t, say N_t and it can be easily shown that for a fixed value of ε, N_t is an increasing function of t. It follows that if we want to compute $\Pr(X(t) \in B_i)$ for M distinct values of t denoted by $t(1) < \cdots < t(M)$ we only need to compute $\Pr(Z(n) \in B_i)$ for $n = 1, \ldots, N_{t(M)}$ since these values are independent of the parameter t.

4 The Busy Period

Let us define the subset B_0' as the subset of states corresponding to at least one customer in the queue, that is

$$B_0' = S - B_0 = \{(n, j, k) \mid n \geq 1,\ 1 \leq j \leq m,\ 1 \leq k \leq l\}.$$

The busy period, denoted by BP, is defined by the first passage time from subset B_0' to subset B_0.

We denote by β an initial probability distribution concentrated on subset B_0', that is $\beta\mathbf{1} = 1$. This distribution can then be written as $\beta = (\beta_h)_{h\geq 1}$ where subvector β_h, which is of dimension ml, corresponds to the initial probability distribution when there are exactly h customers in the queue. The distribution of the busy period BP is then given by

$$\Pr(BP > t) = \beta e^{Gt}\mathbf{1},$$

where matrix G is the submatrix obtained from the infinitesimal generator A by deleting the first m rows and columns which correspond to subset B_0.

In the same way, we define Q as the submatrix obtained from matrix P by deleting the first m rows and columns. Matrix Q is then a substochastic matrix given by the relation $Q = I + G/\nu$, and so we get

$$\Pr(BP > t) = \sum_{n=0}^{\infty} e^{-\nu t}\frac{(\nu t)^n}{n!}\beta Q^n \mathbf{1}.$$

Defining the infinite column vector $U^{(n)}$ as $U^{(n)} = Q^n\mathbf{1}$, we get the recurrence relation $U^{(n)} = QU^{(n-1)}$ with $U^{(0)} = \mathbf{1}$.

Let $V^{(n)}$ be defined by $V^{(n)} = \mathbf{1} - U^{(n)}$. We then get

$$V^{(n)} = V^{(1)} + QV^{(n-1)} \text{ with } V^{(0)} = 0 \text{ and } V^{(1)} = \mathbf{1} - Q\mathbf{1}. \tag{3}$$

We decompose the infinite column vector $V^{(n)}$ into subvectors $V_i^{(n)}$, $i \geq 1$ of dimension ml. For $i \geq 1$, vector $V_i^{(n)}$ can be interpreted using the uniformized Markov chain $Z(n)$ as

$$V_i^{(n)} = [\Pr(NV \leq n | Z(0) = (i, j, k))]_{1\leq j\leq m, 1\leq k\leq l},$$

where NV denotes the number of states visited during the busy period. It follows immediately that for $i \geq n+1$, we have $V_i^{(n)} = 0$. Using relation (3), we obtain the following recurrence relation, using the fact that $P_{1,0}\mathbf{1} = P_0\mathbf{1}$.

$$
\begin{aligned}
V_1^{(n)} &= P_0\mathbf{1} + \sum_{h=1}^{n-1} P_h V_h^{(n-1)} \\
V_i^{(n)} &= \sum_{h=i-1}^{n-1} P_{h-i+1} V_h^{(n-1)} \qquad \text{for } 2 \le i \le n
\end{aligned}
\tag{4}
$$

The matrix P being a substochastic matrix the sequence of vectors $U^{(n)}$ decreases to 0 when n tends to infinity, for $\rho < 1$ where ρ is the traffic intensity. It follows that the computation can be stopped at step N_1 when the number $\beta U^{(N_1)}$ is sufficiently small or equivalently when $\beta V^{(N_1)}$ is sufficiently close to 1. More precisely, for a given error tolerance ε, let us define N_1 as

$$
N_1 = \min\left\{ n \in \mathbb{N} \;\middle|\; \beta V^{(n)} \ge 1 - \varepsilon \right\}. \tag{5}
$$

Using the truncation step N defined by (2), and defining $N' = \min(N, N_1)$, we get

$$
\begin{aligned}
\Pr(BP > t) &= \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \beta U^{(n)} \\
&= 1 - \sum_{n=0}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \beta V^{(n)} \\
&= \sum_{n=0}^{N'} e^{-\nu t} \frac{(\nu t)^n}{n!} - \sum_{n=0}^{N'} e^{-\nu t} \frac{(\nu t)^n}{n!} \beta V^{(n)} - e(N'),
\end{aligned}
$$

where

$$
e(N') = \sum_{n=N'+1}^{\infty} e^{-\nu t} \frac{(\nu t)^n}{n!} \beta U^{(n)} \le \varepsilon.
$$

We will see in the next section that the truncation step N' can be significantly less than N. Note that both Remarks 1 and 2 of the previous section also apply here for the computation of $\Pr(BP > t)$.

5 Numerical Results

We consider a BMAP which is the superposition of h identical Markov Modulated Poisson Processes (MMPPs). Each MMPP alternates between two states: state 0 and state 1. The transition rate from state 0 to state 1 is denoted by μ_0 and the transition rate from state 1 to state 0 is denoted by μ_1. The arrival rate from state 0 (resp. 1) is denoted by λ_0 (resp. λ_1). The auxiliary phase in the overall BMAP can be characterized by the number of MMPPs that are in state 1. This number is initialy supposed to be equal to 0, that is, all the MMPPs are initialy in state 0. The service times distribution is assumed to be Erlang of order l (denoted by E_l, $l \geq 1$) with unit mean time so that the time units are in mean service times

It follows that the overall BMAP is given by the matrices D_0 and D_1 ($D_k = 0$ for $k \geq 2$) of dimension $m \times m$ where $m = h + 1$. The non-zero entries of matrices D_0 and D_1 are

$$\begin{aligned}
D_0(i,i) &= -[(h-i)(\lambda_0 + \mu_0) + i(\lambda_1 + \mu_1)] \text{ for } 0 \leq i \leq h \\
D_0(i,i+1) &= (h-i)\mu_0 \text{ for } 0 \leq i \leq h-1 \\
D_0(i,i-1) &= i\mu_1 \text{ for } 1 \leq i \leq h \\
D_1(i,i) &= (h-i)\lambda_0 + i\lambda_1 \text{ for } 0 \leq i \leq h
\end{aligned}$$

The arrival rate of the overall BMAP is then

$$\lambda = h\left[\lambda_0 \frac{\mu_1}{\mu_0 + \mu_1} + \lambda_1 \frac{\mu_0}{\mu_0 + \mu_1}\right].$$

The mean service times being equal to 1, the traffic intensity ρ is given by $\rho = \lambda$. In all figures we assume that $\lambda_0 = 0.01$, $\lambda_1 = 0.04$, $\mu_0 = 0.04$, $\mu_1 = 0.01$ and the error tolerance is $\varepsilon = 0.0001$. Moreover, all the MMPPs are supposed to be initialy in state 0.

Figure 1 shows the complementary cumulative distribution function of the transient queue length for the $\sum_{r=1}^{20} MMPP_r/E_4/1$ model, which gives a traffic intensity $\rho = 0.68$. The initial queue length is $i' = 0$. It can be noted that for values of t greater than 100 the curves coincide with the curve obtained for $t = 100$ and so the steady state seems to be reached.

Figure 2 shows the complementary emptiness function for the $\sum_{r=1}^{20} MMPP_r/E_4/1$ model which gives a traffic intensity $\rho = 0.68$, for different initial queue length. Note that the steady state value of the emptiness function ($\rho = 0.68$) is reached for $t = 200$.

Figure 3 shows the complementary emptiness function for the $\sum_{r=1}^{20} MMPP_r/E_l/1$ model which gives a traffic intensity $\rho = 0.68$, for different values of the number l of phases in the E_l service distribution. The initial queue length is $i' = 20$. It is interesting to note that for t greater than 80 there are no more differences between the curves and so it seems that the number of service phases does not have a great influence on the emptiness function. Here the steady state value of the emptiness function ($\rho = 0.68$) is reached for $t = 100$.

Figures 4 and 5 show the complementary emptiness function for the $\sum_{r=1}^{h} MMPP_r/E_4/1$ model which gives a traffic intensity $\rho = h \times 0.034$, for different values of the number h of MMPPs. The initial queue length is $i' = 20$ in Figure 4 and $i' = 0$ in Figure 5. Note that for

$h = 29$ we get $\rho = 0.986$, for $h = 30$ we get $\rho = 1.02$, and for $h = 40$ we get $\rho = 1.36$. Note also the convergence to the steady state value $\rho < 1$ for $h \leq 29$, and the convergence to 1 for $h \geq 30$, that is $\rho > 1$.

Figure 6 shows the complementary cumulative distribution function of the busy period for the $\sum_{r=1}^{20} MMPP_r/E_4/1$ model which gives a traffic intensity $\rho = 0.68$, for different values of the initial queue length. Note the rate of convergence to 0 depends on the initial queue length. Concerning the truncation steps, we have for $t = 200$, $N = 1125$ and $N' = 86$ for $i' = 1$, $N' = 269$ for $i' = 5$, $N' = 506$ for $i' = 10$, $N' = 878$ for $i' = 20$, and $N' = N$ for $i' \geq 30$.

Figure 7 shows the complementary cumulative distribution function of the busy period for the $\sum_{r=1}^{20} MMPP_r/E_l/1$ model which gives a traffic intensity $\rho = 0.68$, for different values of the number l of phases in the E_l service distribution. The initial queue length is $i' = 20$. As for Figure 3, it is interesting to note that there are not many differences between the curves and so it seems that the number of service phases does not have a great influence on the busy period distribution. In this case, for $t = 200$ and $l = 6$, we have $N = 1549$ and $N' = 1172$.

Figure 8 shows the complementary cumulative distribution function of the busy period for the $\sum_{r=1}^{h} MMPP_r/E_4/1$ model which gives a traffic intensity $\rho = h \times 0.034$, for different values of the number h of MMPPs. The initial queue length is $i' = 20$.

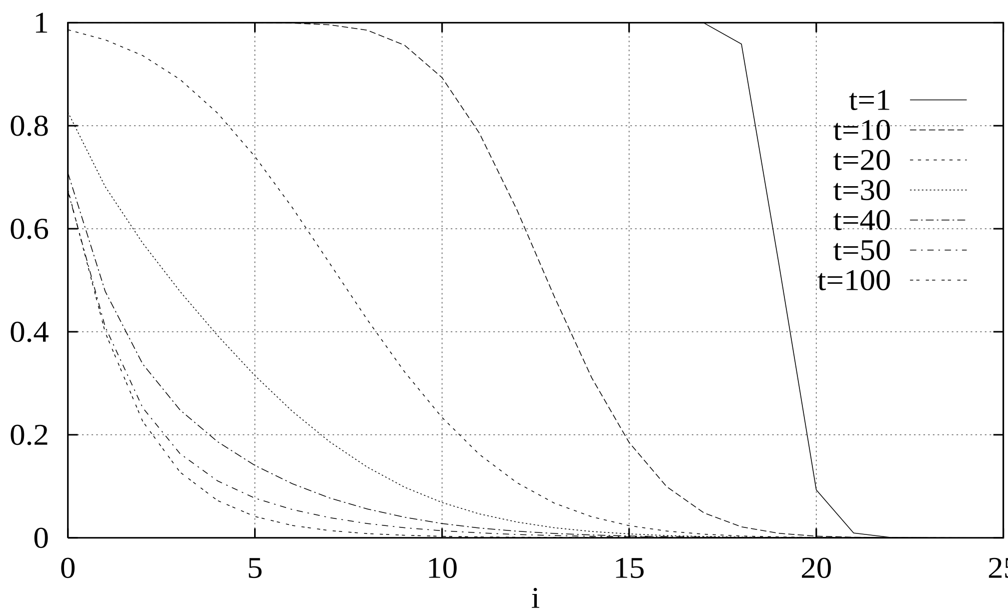

Figure 1: $\Pr(N(t) > i)$, as a function of i and t in the $\sum_{r=1}^{20} MMPP_r/E_4/1$ queue with initial queue length $i' = 20$.

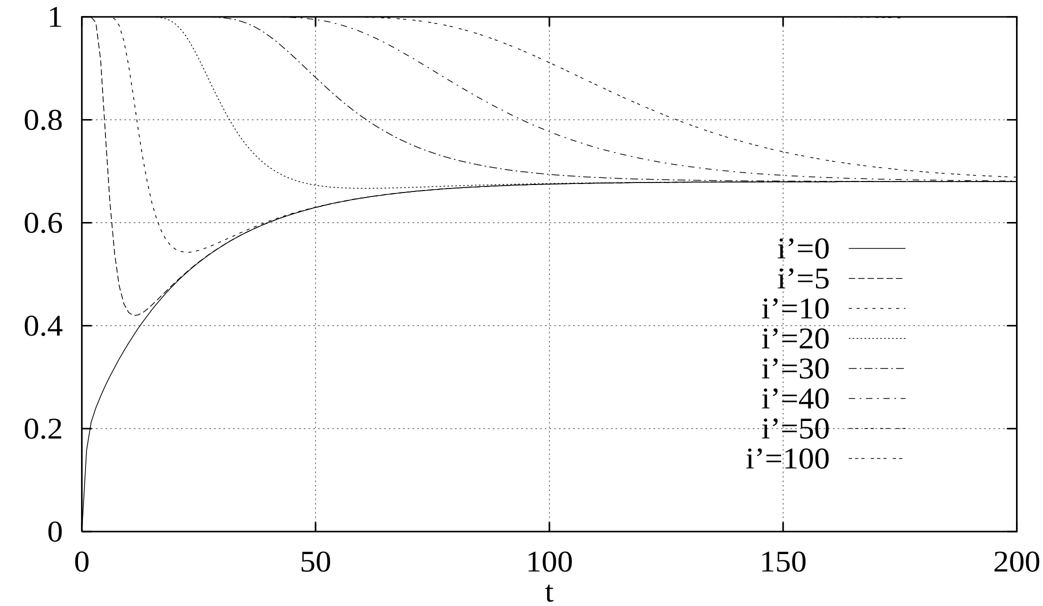

Figure 2: $\Pr(N(t) > 0)$, as a function of t and the initial queue length i' in the $\sum_{r=1}^{20} MMPP_r/E_4/1$ queue.

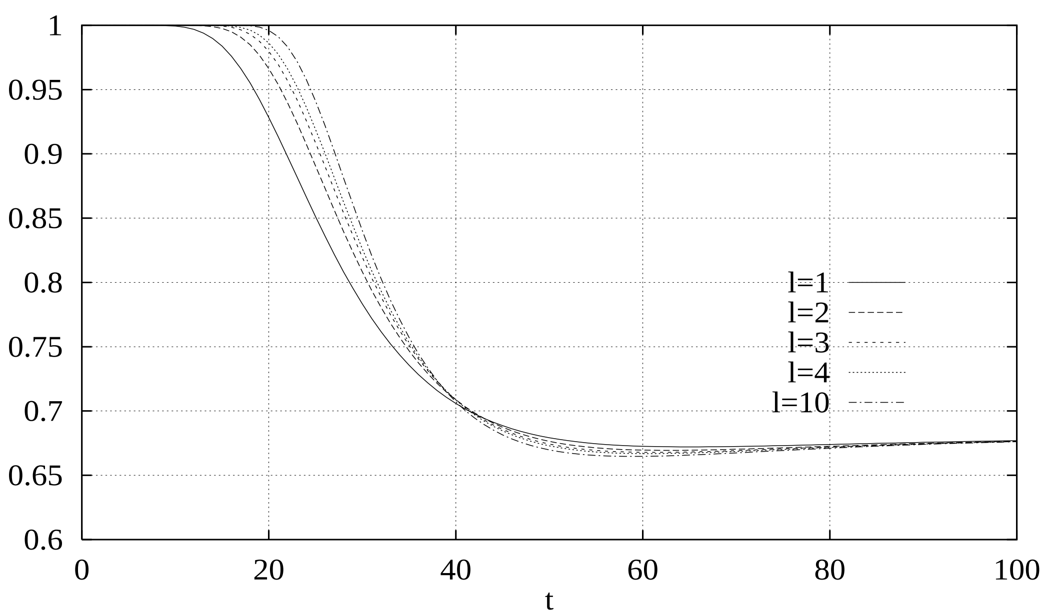

Figure 3: $\Pr(N(t) > 0)$, as a function of t for different values of the number l of phases in the $\sum_{r=1}^{20} MMPP_r/E_l/1$ queue with initial queue length $i' = 20$.

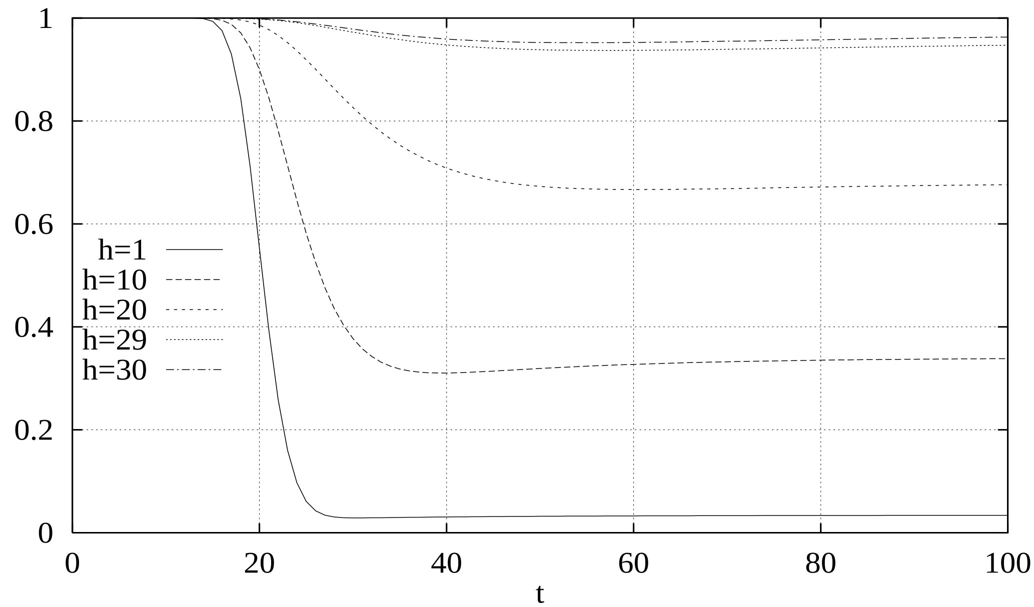

Figure 4: $\Pr(N(t) > 0)$, as a function of t for different values of the number h of MMPPs in the $\sum_{r=1}^{h} MMPP_r/E_4/1$ queue with initial queue length $i' = 20$.

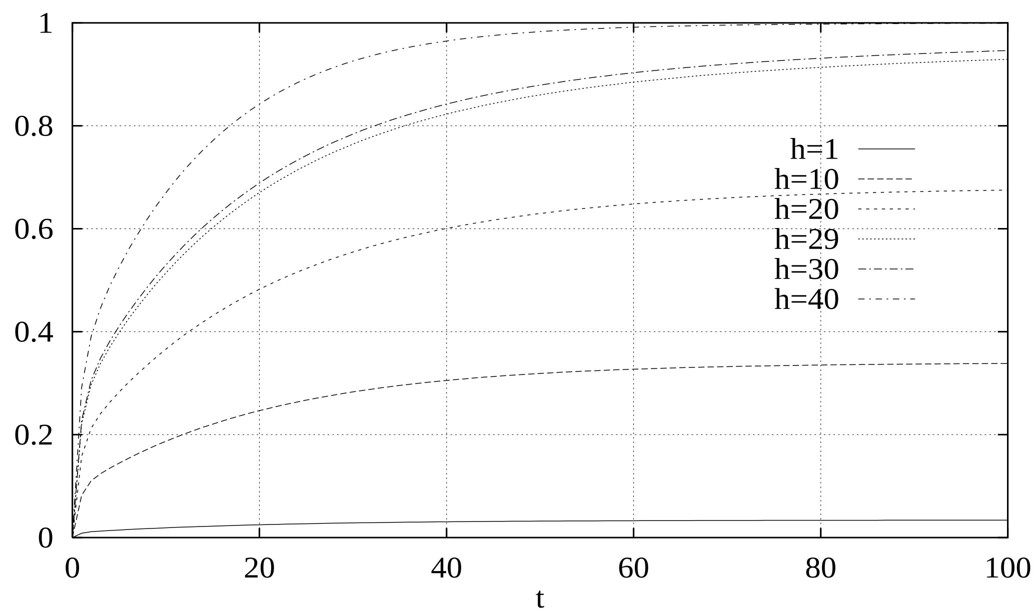

Figure 5: $\Pr(N(t) > 0)$, as a function of t for different values of the number h of MMPPs in the $\sum_{r=1}^{h} MMPP_r/E_4/1$ queue with initial queue length $i' = 0$.

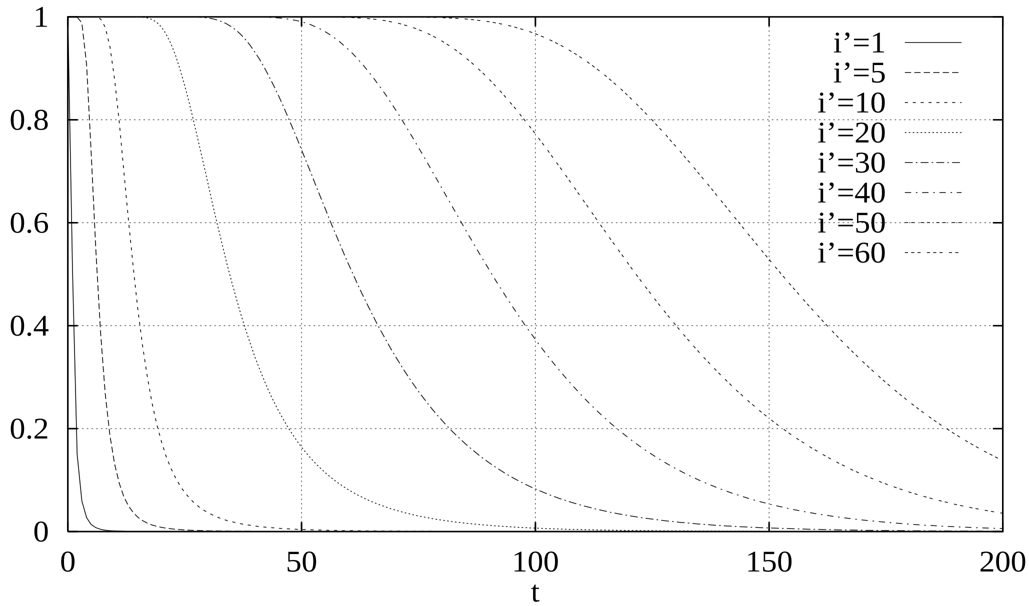

Figure 6: $\Pr(BP > t)$, as a function of t and the initial queue length i' in the $\sum_{r=1}^{20} MMPP_r/E_4/1$ queue.

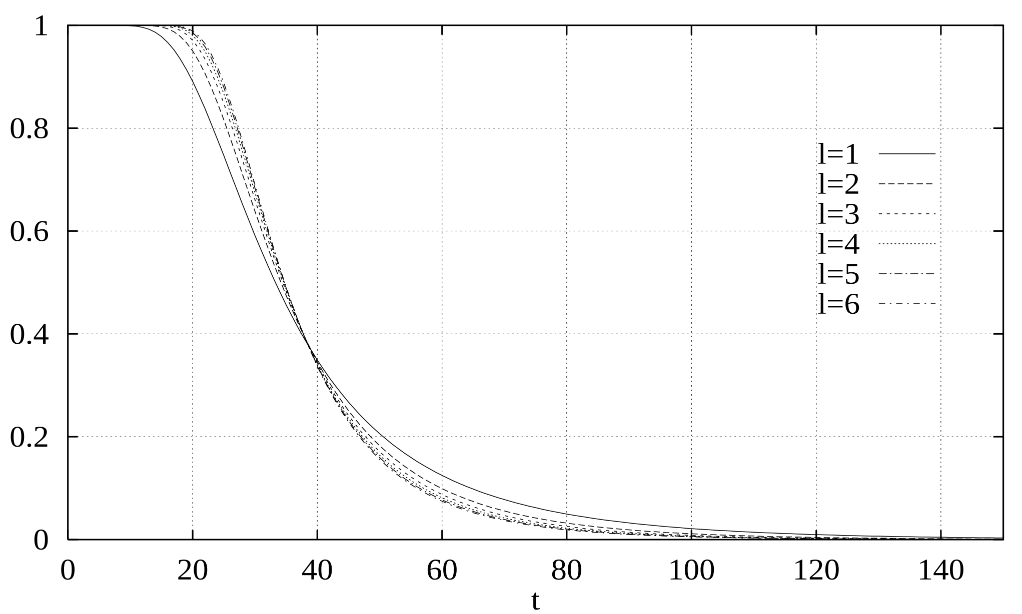

Figure 7: $\Pr(BP > t)$, as a function of t for different values of the number l of phases in the $\sum_{r=1}^{20} MMPP_r/E_l/1$ queue with initial queue length $i' = 20$.

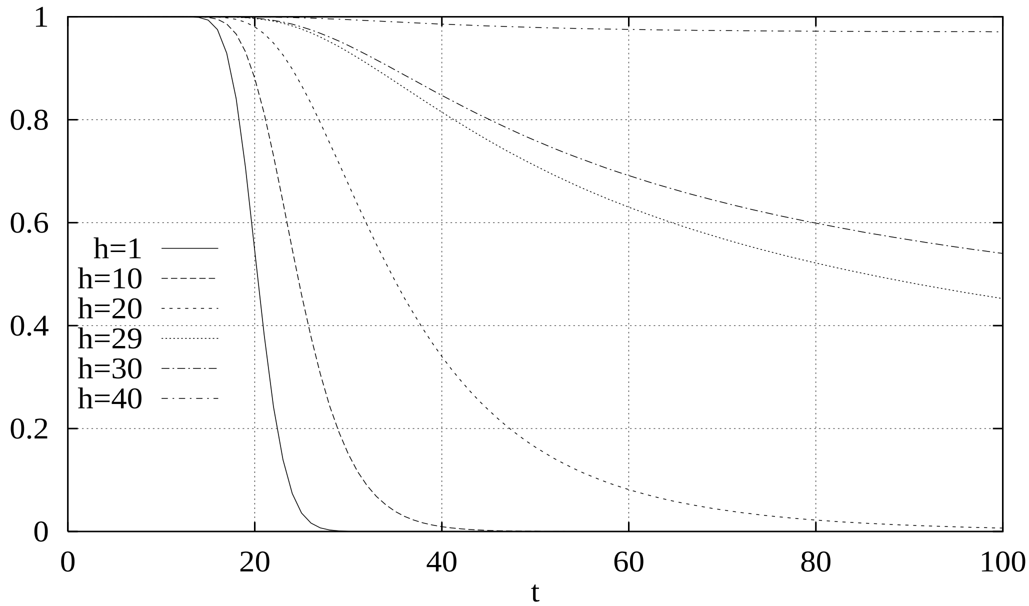

Figure 8: $\Pr(BP > t)$, as a function of t for different values of the number h of MMPPs in the $\sum_{r=1}^{h} MMPP_r/E_4/1$ queue with initial queue length $i' = 20$.

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